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# On a Paper of C. B. Dunham Concerning Degeneracy in Mean Nonlinear Approximation

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In the preceding paper [3], Dunham raised the question whether for $n \geqslant 2$ there exists an $f \notin V_n$,

$$V_n = \left\{ F(A, x) = \sum_{k=1}^{n} a_k \exp(a_{n+k}x) \colon a_k \in \mathbb{R}, k = 1, 2, \ldots, 2n \right\},$$

with a best approximation of degeneracy 2 or more; i.e., a best approximation in $V_{n-2}$. Here, approximation is understood in the sense of $L_1$-norm over a finite real interval $I = [a, b]$. In this note, we will answer this question by constructing a function $f \in C[I]$, $f \not\equiv 0$, for which 0 is the unique best approximation in $V_n$. Note that Dunham gave the opposite answer to the analogous problem for rational functions [4].

Our proof depends heavily on an estimation for the derivative of exponential sums, which is of independent interest. It holds even for the functions in the strong closure of $V_n$.[1]

LEMMA 1. *Let $a < \alpha \leqslant \beta < b$. There exists a positive constant $c = c(n, a, b, \alpha, \beta)$ such that*

$$\max_{x \in [\alpha, \beta]} |g'(x)| \leqslant c \cdot \max_{x \in [a, b]} |g(x)| \tag{1}$$

*for all $g \in \bar{V}_n$.*

Although this lemma has not been stated explicitly in Schmidt's paper [5], it is an immediate consequence of (2.13) in [5]. If $g \in \bar{V}_n$, then we also have $g^{(n+1)} \in \bar{V}_n$, and $g^{(n+1)}$ has at least $n - 1$ zeros or vanishes identically [1].

[1] A representation for the functions in the closure is given in [1, 5]. Observe that $V_n$ in this note corresponds to $V_n^0$ in [1] and to $E_n^0$ in [5], while $\bar{V}_n$ corresponds to $V_n$ and $E_n$, respectively.

Set $K = \max | g(x)|$. Then $g$ belongs to the sets satisfying the assumptions of Theorem 1 in [5]. Hence, that inequality (2.13) may be applied to yield an estimation of the derivative in the subinterval depending only on $n, a, b, \alpha, \beta$, and (linearly) on $K$. Another proof which shows the dependency of $c$ on the parameters will be given in [2].

Now, according to Dunham's paper it is sufficient for our purpose to find a function $f$, satisfying for all $h \in V_n \backslash 0$

$$\left| \int_{I \backslash Z} h \cdot \operatorname{sgn}(f) \right| < \int_Z | h |, \tag{2}$$

where

$$Z = \{x : f(x) = 0, x \in I\}. \tag{3}$$

We choose $\alpha, \beta_0$ satisfying $a < \alpha < \beta_0 < b$. Set $c = c(n + 1, a, b, \alpha, \beta_0)$,

$$\beta = \min(\beta_0, \alpha + 1/3c), \tag{4}$$

and

$$f(x) = \begin{cases} (x - \alpha) \cdot (\beta - x), & \text{if } \ x \in (\alpha, \beta), \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Obviously, we have $Z = [a, \alpha] \cup [\beta, b] \neq I$. Let $h \in V_n$, $h \not\equiv 0$. Then

$$g(x) = \int_a^x h(y)\, dy \in \bar{V}_{n+1} .$$

Since

$$| g(x)| \leqslant \int_a^x | h(y)|\, dy \leqslant \int_I | h |$$

for $x \in [a, b]$, it follows from Lemma 1 that

$$| h(x)| = | g'(x)| \leqslant c \cdot \sup_{y \in I} | g(y)| \leqslant c \int_I | h |, \qquad \alpha \leqslant x \leqslant \beta, \tag{6}$$

where $c$ is the constant used in the construction of $f$. By integrating (6), we obtain

$$\int_\alpha^\beta | h(x)|\, dx \leqslant \frac{1}{3c} c \int_I | h | < \frac{1}{2} \int_I | h | = \frac{1}{2} \int_\alpha^\beta | h | + \frac{1}{2} \int_Z | h |.$$

Hence

$$\left| \int_{I \backslash Z} h \cdot \operatorname{sgn}(f) \right| \leqslant \int_\alpha^\beta | h | < \int_Z | h |,$$

which completes the proof.

## References

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